A simple Robinson-Crusoe economy

Consider Robinson-Crusoe sitting on his island who is trying to survive by either gathering bananas (x denotes the number of bananas) or going fishing (y is the number of fish).

■ 1. Technologies

The technologies which are available for these two production processes are given by [notice that this set-up excludes such interesting issues like learning-by-doing (human capital accumulation), the search for new and more efficient technologies (technological progress) or building helpful devices (investment in physical capital)]

$$x = \sqrt{Lx} \qquad \Longleftrightarrow \qquad x^2 = Lx$$

$$y = 0.5\sqrt{Ly} \qquad \Longleftrightarrow \qquad 4\,y^2 = Ly$$

where L_x is the amount of labour (time) used for gathering bananas and L_y the time devoted to fishing. Notice that there are diminishing returns in both activities (e.g. due to limited supply of the natural goods under consideration). Total time (L) is constrained to 15 hours per day.

$$L = Lx + Ly = 15$$

From the information given above, we can construct the production possibility frontier (PPF). An implicit form for the PPF is

$$L = x^2 + 4\,y^2$$

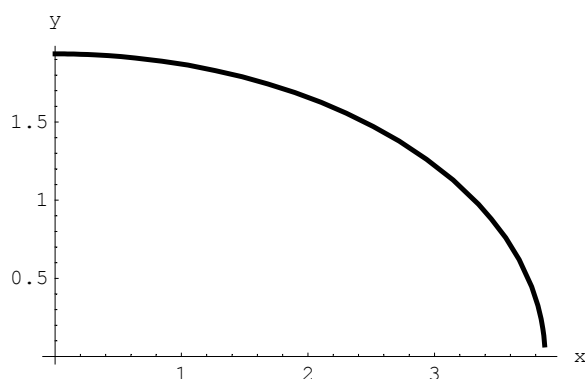Solving for y yields the PPF in explicit form

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Solve[L == x^2 + 4 y^2, y]
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$$\left\{\left\{y \to -\frac{1}{2}\sqrt{L - x^2}\right\}, \left\{y \to \frac{\sqrt{L - x^2}}{2}\right\}\right\}$$

We exclude the negative solution since this does not make any sense economically. Hence the PPF is given by

$$y = \frac{\sqrt{L - x^2}}{2}$$

Using the Plot command we can readily illustrate the shape of the PPF

```
L = 15; ppf = Plot[ (√(L - x²))/2 , {x, 0, 5}, PlotStyle → Thickness[0.01], AxesLabel → {x, y}];
```


The marginal rate of transformation (MRT) is the slope of the PPF. Letting $y = y(x)$ denote the PPF, the MRT is given by

$$\text{MRT} = \frac{\partial y(x)}{\partial x}$$

The MRT can be determined conveniently by using the D command

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MRT = D[ (√(L - x²))/2 , x]
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$$- \frac{x}{2 \sqrt{15 - x^2}}$$

Interpretation: Assume that Robinson-Crusoe produces 2 units of x initially. Evaluating the MRT at $x = 2$ gives

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MRT /. x → 2 // N
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-0.301511
```

If Robinson Crusoe wants to produce one more unit of x, then he is forced to reduce the number of y by 0.3.

■ 2. Preferences

We assume that Robinson Crusoe's preferences are described by a Cobb-Douglas utility function

$$U(x, y) = \sqrt{x} \ \sqrt{y}$$

The marginal rate of substitution (MRS) is given by

$$\text{MRS} = \frac{dy}{dx} = -\frac{\partial U(x, y)/\partial x}{\partial U(x, y)/\partial y}$$

Note: The MRS can be derived from totally differentiating utility and setting the result equal to zero; $dU = \frac{\partial U(.)}{\partial x} \ dx + \frac{\partial U(.)}{\partial y} \ dy = 0$ gives $\frac{dy}{dx} = -\frac{\partial U(.)/\partial x}{\partial U(.)/\partial y}$.

The MRS can be readily found to read as follows

```
In[1]:=  U = √x √y ;
```

$$In[2]:= \quad -\frac{D[U, x]}{D[U, y]}$$

$$Out[2]= \quad -\frac{y}{x}$$

Interpretation: Assume Robinson Crusoe consumes initially 1 unit of y and 2 units of x; hence MRS = 0.5. Reducing the consumption of y by 0.1 units requires to increase the consumption of x by 0.2 units to keep utility constant. The MRS hence gives Robinson's marginal valuation of good y in terms of good x (and vice versa).

To illustrate, we can plot a number of indifference curves for alternative utility levels

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ContourPlot[√x √y , {x, 0, 3}, {y, 0, 2}, ContourShading → False];
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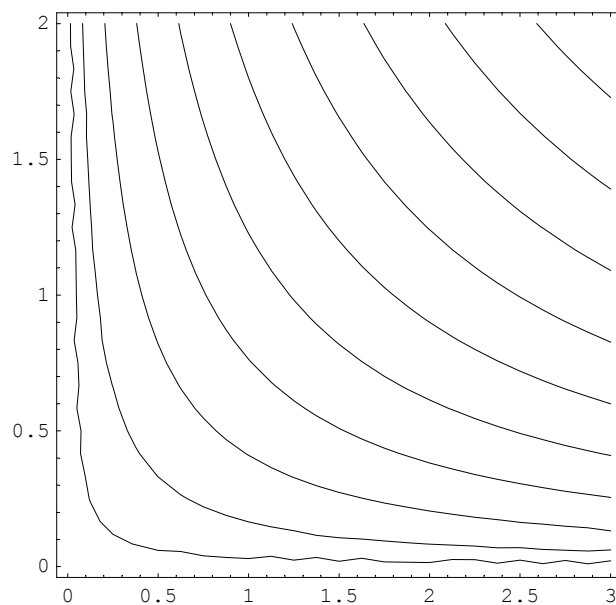

■ 3. Robinson's equilibrium

How does Robinson Crusoe allocate his scarce time of 15 hours per day on the two activities "banana gathering" and "fishing" provided that he behaves rational in sense of maximizing utility? From economic theory, we know that utility is maximized if MRT = MRS. We employ this efficiency condition to solve the underlying problem.

For convenience, the MRT and MRS are restated below

$$MRT = -\frac{x}{2\sqrt{15 - x^2}}$$

$$MRS = -\frac{y}{x}$$

From MRT = MRS we get

$$\frac{x}{2\sqrt{15 - x^2}} = \frac{y}{x}$$

At this point we have two equations in x and y, which are the PPF and the equation resulting from MRS = MRT. From these two equations we can determine the optimal values of x and y. The PPF and the equation resulting from MRS = MRT are restated for convenience

$$y = \frac{\sqrt{L - x^2}}{2} \qquad \text{(PPF)}$$

$$\frac{x}{2\sqrt{15 - x^2}} = \frac{y}{x} \qquad \text{(MRS = MRT)}$$

Using the Solve command, we can readily determine the optimal amount of x (bananas) and y (fish) to read as follows

```
L = 15;
```

$In[5]:=$ $\texttt{Solve}\left[\left\{\texttt{y} == \frac{\sqrt{\texttt{L} - \texttt{x}^2}}{2}, \frac{\texttt{x}}{2\sqrt{15 - \texttt{x}^2}} == \frac{\texttt{y}}{\texttt{x}}\right\}, \{\texttt{x}, \texttt{y}\}\right]$

$Out[5]=$ $\left\{\left\{\texttt{y} \to \frac{\sqrt{\frac{15}{2}}}{2}, \texttt{x} \to -\sqrt{\frac{15}{2}}\right\}, \left\{\texttt{y} \to \frac{\sqrt{\frac{15}{2}}}{2}, \texttt{x} \to \sqrt{\frac{15}{2}}\right\}\right\}$

Only positive amounts make sense economically and hence the negative solution is excluded.

To determine the level of utility, the utility function is simply evaluated at $x = x^*$ and $y = y^*$

$In[6]:=$ $\texttt{U} = \sqrt{\texttt{x}} \sqrt{\texttt{y}} /. \left\{\texttt{x} \to \sqrt{\frac{15}{2}}, \texttt{y} \to \frac{\sqrt{\frac{15}{2}}}{2}\right\}$
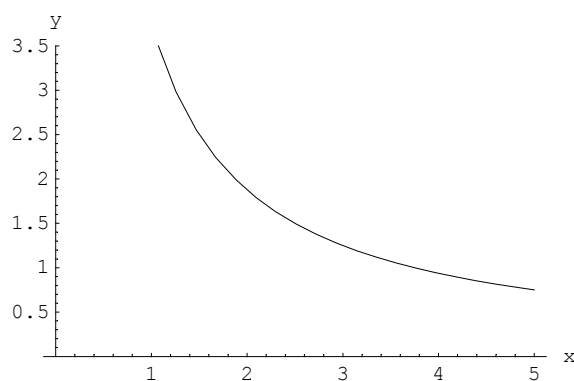
$Out[6]=$ $\frac{\sqrt{15}}{2}$

One can plot the indifference curve which is associated with this maximal level of utility. To do this we set $\frac{\sqrt{15}}{2} = \sqrt{x} \sqrt{y}$, solve for y and plot the resulting expression over x

$\texttt{Solve}\left[\frac{\sqrt{15}}{2} == \sqrt{\texttt{x}} \sqrt{\texttt{y}}, \texttt{y}\right]$

$\left\{\left\{\texttt{y} \to \frac{15}{4\texttt{x}}\right\}\right\}$

$\texttt{maxutility} = \texttt{Plot}\left[\frac{15}{4\texttt{x}}, \{\texttt{x}, 0, 5\}, \texttt{PlotRange} \to \{0, 3.5\}, \texttt{AxesLabel} \to \{\texttt{x}, \texttt{y}\}\right];$

Combining the plot of the PPF and the maximum indifference curve gives a nice graphical illustration of the equilibrium allocation

Show[maxutility, ppf];
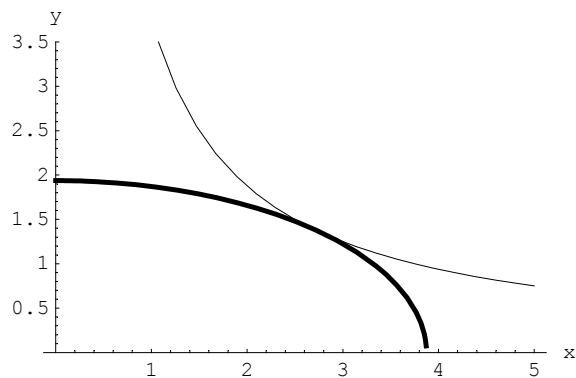
Finally, we show the slope of the PPF (MRT) and the slope of the indifference curve (MRS) at the optimal point.

MRT /. x -> $\sqrt{\dfrac{15}{2}}$

$-\dfrac{1}{2}$

To plot the line which runs through the equilibrium point we have to determine the following equation $y = a + b\,x$. The slope is known $b = 0.5$. The absolute value a can be determined from $x^* = \sqrt{\frac{15}{2}}$ and $y^* = \frac{\sqrt{\frac{15}{2}}}{2}$.
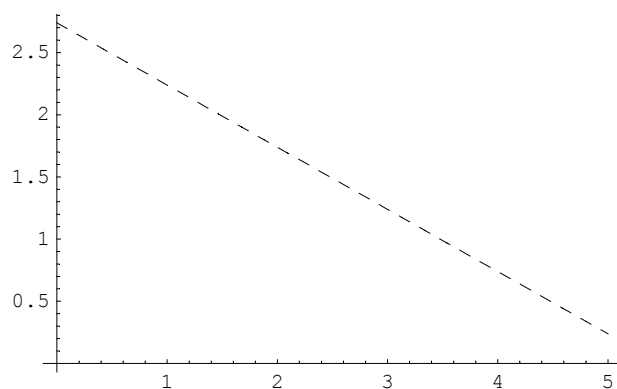
equation = y == a + b x /. $\left\{ x \to \sqrt{\dfrac{15}{2}} \, , \, y \to \dfrac{\sqrt{\frac{15}{2}}}{2} \, , \, b \to -0.5 \right\}$;
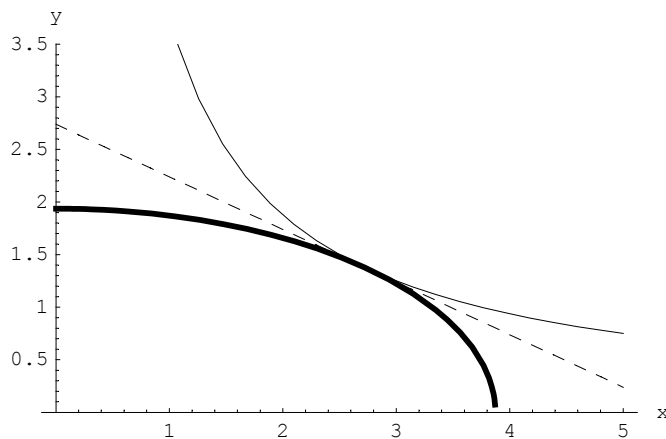
Solve[equation, a]

$\{\{a \to 2.73861\}\}$

Now we can plot the straight line which runs through the optimal solution (x^*, y^*) and gives the MRT = MRS.

impliciteprice = Plot[2.7386 - 0.5 x, {x, 0, 5}, PlotStyle → Dashing[{0.02}]];

Combining all three plots finally yields

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equilibrium = Show[maxutility, ppf, impliciteprice];
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■ 4. Basic interpretation

The Robinson-Crusoe story is very useful because it exemplifies the basic economic decisions faced by a consumer-producer household. Moreover, this basic model can be readily extended into several directions: (1) A second individual enters the stage (Friday); (2) endogenous "labour supply"; (3) learning-by-doing effects; (4) search for new and better technologies (R&D); (5) investment in physical capital; or (6) uncertainty.

Economists do not assert that "Robinson Crusoe" actually sits down and calculates the optimal solution. Indeed this would probably a waste of scarce resources (time). What economic theory suggests is this: In order to understand the behavior of Robinson Crusoe (positive theory) the underlying model can be fairly helpful. Specifically, it appears reasonable to assume that Robinson Crusoe behaves rational in the sense of maximizing utility subject to scarcity and technological constraints. The decisions taken are then determined by preferences and technology.